The Condition of Forming the Bose Einstein Condensation in an External Potential

Yun Gu

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Department of Physics, School of Science, Tianjin University, 92 Wenjin Road, Tianjin 300072, China

1 Introduction

With the development of techniques for trapping atoms, we have a chance to observe the Bose-Einstein condensation (BEC) experimentally. Normally, we consider a non-interacting Bose gas with mass m in a rigid container of volume V. In this condition, BEC occurs only when the dimensionality of the space is more than 2.
However, if the atoms are confined by an external potential of the form $V(r) = \alpha r^s$, rather than a rigid container potential, the possibility of forming the BEC will depend on both the dimensionality and the form of the potential. We will present the theoretical results for the relationship between dimensionality (d) and the form of the potential.

2 When the BEC Will Happen

As is well known, the density of states $D(\epsilon) \propto \epsilon^p$ (since we are considering a thermodynamic problem, we can assume $D(\epsilon) \propto \epsilon^p$). The exponent of the density of states, which is actually p, will dominate when the BEC will happen. So, we calculate the particle number located in the state $\epsilon > 0$ ($N_{\epsilon=0}$),

$$N_{\epsilon>0} = \int_0^\infty d\epsilon D(\epsilon) \frac{1}{\exp\{\beta\epsilon\} - 1} \tag{1}$$

$$\propto \int_0^\infty d\epsilon \epsilon^p \frac{1}{\exp\{\beta\epsilon\} - 1} \tag{2}$$

When $N_{\epsilon>0}$ is finite, the particle in state $\epsilon = 0$ ($N_{\epsilon=0}$) will have the same order of magnitude as $N_{\epsilon>0}$. Therefore, the BEC will occur.

We consider equation 2. The integral converges when $\epsilon \to \infty$. And when $\epsilon \to 0$, the integrand function is:

$$\frac{\epsilon^p}{\exp\{\beta\epsilon\} - 1} \sim \frac{\epsilon^p}{\beta\epsilon} = \frac{1}{\beta}\epsilon^{p-1} \tag{3}$$

The convergence condition of the integral is $p - 1 > -1 \Rightarrow p > 0$. So the condition for forming the BEC is $p > 0$.

3 The Form of Density of State in an External Potential

We will evaluate the form of the density of state with the canonical partition function. So we can figure out the important p in the density of state. Actually, it is a dynamic problem, but it's convenient to solve the problem using this thermodynamic method (using the canonical partition function). If we obtain the relation function of $p(s, d, \alpha)$ (d is dimensionality, s, α is the index in the external potential $V(r) = \alpha r^s$), we will solve the condition of forming the BEC in an external potential.
We can write down the canonical partition function (note $Z(\beta)$) in two equivalent ways.

$$Z(\beta) \propto \int_0^\infty \exp\{-\beta\epsilon\} D(\epsilon) d\epsilon \tag{4}$$

$$Z(\beta) \propto \int_0^\infty \exp\{-\beta H(p, q)\} d\mathbf{p} d\mathbf{q} \tag{5}$$

Equation 4 can be simplified as:

$$Z(\beta) \propto \int_0^\infty \exp\{-\beta\epsilon\} D(\epsilon) d\epsilon = \int_0^\infty \exp\{-\beta\epsilon\} \epsilon^p d\epsilon \propto \beta^{-p-1} \tag{6}$$

So, equation 5 will be simplified as:

$$
\begin{aligned}
Z(\beta) &\propto \int_0^\infty \exp\{-\beta H(\mathbf{p},\mathbf{q})\} d\mathbf{p} d\mathbf{q} \\
&\propto \int_0^\infty \exp\{-\beta\{\frac{\mathbf{p}^2}{2m} + \alpha r^s\}\} d\mathbf{p} d\mathbf{q} \\
&\propto \int_0^\infty \exp\{-\beta\{\frac{p^2}{2m} + \alpha r^s\}\}(p^{d-1}dp)(r^{d-1}dr) \\
&\propto \int_0^\infty \exp\{-\beta\frac{p^2}{2m}\} p^{d-1}dp \times \int_0^\infty \exp\{-\beta\alpha r^s\} r^{d-1} dr \\
&\propto \beta^{-\frac{1}{2}d} \times \beta^{-\frac{d}{s}} = \beta^{-\frac{1}{2}d-\frac{d}{s}} \qquad (7)
\end{aligned}
$$

Equations 6 and 7 must have the same index of β. So, $-p-1 = -\frac{1}{2}d - \frac{d}{s} \Rightarrow p = \frac{1}{2}d + \frac{d}{s} - 1$.

4 The Final Result

In Section 2, we know the condition for forming BEC is $p > 0$. And in Section 3, we find the relation between d, s, and p. So, the result is when $\frac{1}{2}\mathbf{d} + \frac{d}{s} > \mathbf{1}$, the BEC will exist.